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Dynamics of sandpiles: Physical mechanisms, coupled stochastic equations, and alternative universality classes

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We present a set of coupled nonlinear stochastic equations in one space dimension, designed to model the surface of an evolving sandpile. These include nonlinear couplings to represent the constant transfer between relatively immobile clusters and mobile grains, incorporate the presence of tilt, and contain representations of inertia and evolving configurational disorder. The critical behavior of these phenomenological equations is investigated numerically. It is found to be diverse, in the sense that different combinations of noise as well as different symmetries lead to nontrivial exponents. In the cases most directly comparable with previous studies, we find that our equations lead to a surface with a roughness exponent $\alpha^{\rm tilt}\approx 0.40$, to be compared with the Edwards-Wilkinson and Kardar-Parisi-Zhang values, namely $\alpha^{\rm EW}=\frac{1}{4}$ and $\alpha^{\rm KPZ}=\frac{1}{3}$, respectively. This is, in our view, directly due to the effect of the tilt term. Finally we discuss our results, as well as possible modifications to our equations.

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I. INTRODUCTION

There has been a great deal of recent interest in the physics of sandpiles [1–3], and the complexity of these systems has made them a subject of deep fascination for theoretical physicists in particular. However, well before the current upsurge of interest in sandpiles, there were attempts to model evolving interfaces, such as those in colloidal aggregates or solidification fronts [4]. In all these models, the basic picture was of particle deposition on a rough surface, described by a height $h(\mathbf{x},t)$. The growth of the interface in response to the rearrangement or amplification of local heights was modeled to varying degrees of complexity via Langevin equations for the height, with the noise term representing the effect of the external perturbation.

Most of these models exhibit critical behavior, in the sense that their long-distance physics obeys scaling laws, often referred to as self-organized criticality (SOC) [5]. Three critical exponents α, β, and z characterize the spatial and temporal behavior of a rough interface. They are conveniently defined by considering the connected two-point correlation function of the heights, namely $G(\mathbf{x}-\mathbf{x}',t-t')=\langle h(\mathbf{x},t)h(\mathbf{x}',t')\rangle-\langle h(\mathbf{x},t)\rangle\langle h(\mathbf{x}',t')\rangle$. We have

$$G(\mathbf{x},0)\sim|\mathbf{x}|^{2\alpha}\quad(|\mathbf{x}|\to\infty),$$
$$G(\mathbf{0},t)\sim|t|^{2\beta}\quad(|t|\to\infty), \tag{1.1}$$

and, more generally

$$G(\mathbf{x},t)\approx|\mathbf{x}|^{2\alpha}F(|t|/|\mathbf{x}|^{z}) \tag{1.2}$$

in the whole long-distance scaling regime ($\mathbf{x}$ and t large). The scaling function F is universal; α and $z=\alpha/\beta$, respectively, are referred to as the roughness exponent and the dynamical exponent of the problem.

The first of these approaches was due to Edwards and Wilkinson [6] (EW), and involved a purely diffusive mechanism for the relaxation of the surface. This linear problem is easily solved in any space dimension d. It is critical for $d\leq 2$, where $\alpha=2\beta=1-d/2$, and $z=2$. Kardar, Parisi, and Zhang [7] suggested that a form of the Burgers equation [8] was a more appropriate representation, and suggested that the lowest order nonlinear term to be added to the EW equation was a term $(\nabla h)^2$ representing lateral growth. The full Kardar-Parisi-Zhang (KPZ) equation is recalled hereafter in Eq. (3.1); the EW equation is obtained by setting $g=0$ in the KPZ equation.

The solution of the KPZ equation has been an ongoing problem in theoretical physics, which has been tackled by means of a wide variety of approaches, overviewed in the recent reviews [9,10]. Its critical exponents seem to be nontrivial in any dimension. They are known exactly in

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$1+1$ dimensions: $\alpha=\frac{1}{2}$, $\beta=\frac{1}{3}$, and $z=\frac{3}{2}$, but only approximately in higher dimensions; they obey the relation $\alpha+z=2$. Successful attempts to obtain crossovers away from EW and KPZ equations have been rare, and it has usually been necessary to add complicated terms in order to achieve this; an example of this can be found in the work of Maritan *et al.* [11], which comprises relativistic invariance under reparametrization and leads to a crossover away from KPZ exponents in the long-time limit.

More recently, there have been attempts directed specifically at understanding sandpile avalanches; however, these approaches start from general considerations of symmetry rather than from specific physical considerations germane to sandpiles. Examples of such approaches are those due to Hwa and Kardar [12] and Grinstein and Lee [13]; the approach of Toner [14] is built on the latter to include a representation of disorder.

All the above approaches are unified by the fact that they involve only one variable, the local height of the surface $h(\mathbf{x},t)$, rather than any form of coupling between the moving grains and the relatively immobile clusters which has been shown [15] to be necessary for the understanding of the sandpile surfaces. This coupling was included by Mehta, Needs, and Dattagupta [16], albeit at a relatively macroscopic level in the first instance; the coordinates in the resulting Langevin equations were respectively the macroscopic angle of tilt θ (which is affected by the motion of independent particles) and an average roughness of slope ϕ with respect to this (which represents the average extent to which clusters protrude from the surface). This theory was able to provide a realization of dilatancy [17] and hysteresis, to interpolate via effective temperatures between different dynamical regimes, and provided good agreement with experiment [18].

On the other hand, the role of evolving configurational disorder was investigated by Mehta and Barker [19] via a cellular-automaton model; this work emphasized the need for a representation of an exchange between clusters and grains in any realistic microscopic models of sandpiles, demonstrated, via a simple model of cluster reorganization, the link between surface and bulk in avalanche formation, and finally showed the hysteretic behavior characteristic of these systems. Concurrently, another sandpile cellular automaton designed by the same authors [20] to model a sandpile subjected to constant tilt in a rotating cylinder showed the effects of dissipation and inertia, and demonstrated the (intuitively obvious) role of tilt in generating moving grains from erstwhile frozen clusters.

All these ingredients were included in a more local approach to sandpile dynamics, still containing the crucial nonlinear coupling between moving clusters and grains mentioned above; the equations representing this were first presented in Ref. [1], and their analysis forms the subject of this paper. The effective coordinate representing clusters at a microscopic level is clearly the local height $h(\mathbf{x},t)$, since the geometric fluctuations of clusters are manifested by variations in $h(\mathbf{x},t)$; this is related to the variable ϕ in earlier work [16], while the macroscopic slope in that work has a more microscopic incarnation here in terms of a local density $\rho(\mathbf{x},t)$ of mobile particles. In a sandpile submitted to stirring, shaking, or pouring, these coordinates will clearly be coupled. In Ref. [1] we introduced a particular set of terms to reflect the physical processes at work in the grain-cluster coupling [see Eqs. (2.2) and (2.3) below]. Later, another group [21] analyzed a special case of our equations; these were obtained by setting D_h, λ, and ν to zero in Eqs. (2.2) and (2.3) below, and consequently contained different physics. This difference is mirrored in their results which do not, among other things, include crossovers to regimes characterized by new and nontrivial critical exponents.

In this paper we analyze our original equations, as well as several generalizations of them, to incorporate the effects of different symmetries and different noise terms. The plan of this paper is as follows: we present our dynamical equations in Sec. II, and discuss the various terms in them. In Sec. III, we discuss the effect of differing physical constraints and their accompanying modifications to the form of our equations, and quantify all this with numerical evidence, concerning especially their critical behavior. Finally, in Sec. IV we discuss our results.

II. GENERALITIES

The stochastic dynamical equations investigated in the present work are a generalization of those introduced in Ref. [1]. They concern a one-dimensional sandpile, described by two coupled variables as follows: $h(x,t)$ represents the profile of relatively immobile material, measured from a fixed negative critical slope $-p_c$. In other words the actual height of immobile particles above a horizontal reference line reads

$$y(x,t)=-p_c x+h(x,t) , \tag{2.1}$$

whereas $\rho(x,t)$ represents the density of the layer of moving particles.

The dynamical equations have the following general form:

$$\partial h/\partial t=D_h\nabla^2 h-T+\eta_h(x,t) , \tag{2.2a}$$

$$\partial\rho/\partial t=-\nabla j+D_\rho\nabla^2\rho+T+\eta_\rho(x,t) . \tag{2.2b}$$

For the time being, and along the lines of Ref. [1], we set

$$T=-\kappa\rho\nabla^2 h-\lambda\rho(\nabla h)_+-\mu\rho\nabla h-\nu(\nabla h)_- \tag{2.3}$$

and

$$j=-\gamma\rho(\nabla h)_- . \tag{2.4}$$

We have introduced the notations $\nabla=\partial/\partial x$, and

$$z_+=\begin{cases} z & \text{for } z\geq 0\\ 0 & \text{for } z\leq 0,\end{cases} \quad z_-=\begin{cases} 0 & \text{for } z\geq 0\\ z & \text{for } z\leq 0 .\end{cases} \tag{2.5}$$

The content of our dynamical equations is as follows.

(i) The first term on the right hand side of Eq. (2.2a) is an EW term and represents the rearrangement of clusters in the presence of an applied noise; the associated coupling D_h is a diffusivity. This corresponds to collective relaxation [15], denotes strictly intracluster motion, and exists in the quasistatic as well as grain-inertial regimes [1]. Although this term does not lead to dramatic changes in the analysis of our equations, its physical importance cannot be underestimated. The absence of this term would lead to an unphysical situation in the limit $T=0$, when there are no flowing grains; the clusters would be configurationally frozen, and a vibrated sandpile (finite noise in h) would have no means of relaxation. This is clearly inappropriate, given that clusters have been observed to rearrange [1–3] under conditions of low-intensity vibration *even when* no mobile grains flow down the pile; in fact this process of cluster rearrangement at low vibrational intensities provides a mechanism for collective relaxation, and has been shown [15,22] to be crucial for the compaction of a pile to high densities.

(ii) The second block of terms, T, represents the transfer of clusters to flowing grains, and vice versa. This transfer process has been shown to be crucial [19,20] to sandpile dynamics, and includes representations of evolving disorder and inertia. Included in T are the following terms.

(a) The term $\kappa\rho\nabla^2 h$ represents intercluster motion initiated by the flowing grains. The essential difference between this term and the diffusive term for h is that, whereas this mechanism stops in the absence of avalanches, the diffusive process continues even in this limit. We visualize this as being due to a current of grains moving down the slope, knocking out bumps and filling in holes, so that it is a term that can exist in the grain-inertial regimes but not in the quasistatic one. Note that this is a representation of inertia [20], since it is a mechanism for amplifying sandpile avalanches *independently of slope.*

(b) The term $\mu\rho\nabla h$ also represents intercluster motion of grains, which is mediated by the motion of grains moving independently of each other down the slope [15]. Thus a current of grains moves down the slope, accumulating at points of low slope and knocking out grains from regions of large slope. Also, where as $\kappa\rho\nabla^2 h$ smooths out bumps and dips on the pile irrespective of slope, the $\mu\rho\nabla h$ term exists purely to smooth out deviations from the critical slope. This slope dependence allows us to view this term as a representation of evolving configurational disorder and memory [19]; thus an overly bumpy section of this pile (large deviation from critical slope) will lead to a large avalanche even if a very few grains (small ρ) hit it.

(c) The term $\nu(\nabla h)_-$ represents the spontaneous generation of flowing grains whenever the local slope is larger than critical; this exists even in the absence of flowing grains and is meant to represent the effect of tilting a stationary sandpile. Note that, because we have chosen the critical slope to be negative by definition, a negative sign of ∇h implies that the overall slope is steeper than critical. This term is a simple representation of a crucially important effect in sandpile dynamics, and, as we will see below, is in large part responsible for the novelty of our model from the viewpoint of critical phenomena. However, we emphasize here that in its absence a noiseless (undeposited) pile subjected to slow tilt would stay frozen. Its physics is therefore crucial, in that it is able to reproduce the everyday phenomenon whereby an originally static pile, when tilted, generates mobile (flowing) grains.

(d) The term $\lambda\rho(\nabla h)_+$ can be viewed as a crude representation of the effect of the boundary layer (whose width represents the maximal range for cluster-grain exchange to occur) [1] since its action is to limit the release of flowing grains generated by the effect of tilt; it thus acts as a regulator on the generation of mobile grains. While there is in principle no restriction on the amount of ρ (up to the size of the pile) that can be generated in a real sandpile subjected to tilting or applied noise, we restrict ourselves to the situation where tilting is a moderate perturbation, and only grains occupying the boundary layer can be liberated to flow down the pile via the h-ρ conversion in the tilt term. After a short transient the system relaxes to a stationary state where $\langle \partial\rho/\partial t\rangle=0$. We thus have $\langle\rho\rangle\sim\nu/\lambda$ finite at saturation. This makes quantities such as h and ρ, as well as their fluctuations, finite and measurable, which is essential for the numerical simulations described hereafter. Hence λ plays the role of an experimental (and numerical) cutoff: we replicate existing experimental approaches [3] where vibration and/or tilt are perturbations rather than catastrophes, and it is mainly the boundary layer that is affected by these [2].

(iii) The first term in Eq. (2.2b), $-\nabla j$, represents the variation in ρ due to the nonuniformity of the current of flowing grains, in such a way that the total number of particles is conserved. The current $j(x,t)$ is proportional to the number of mobile grains and to their velocity; the latter is assumed to vanish for slopes greater than critical, and to be proportional to the driving field which is the magnitude of the local slope, from which we derive expression (2.4), the associated parameter γ being a mobility.

(iv) The second term in Eq. (2.2b), $\nabla^2\rho$, represents the relaxation of the flowing grains, and is a crude way of representing intergrain collisions. The coupling D_ρ is again a diffusivity.

(v) Finally, to a discussion of the source terms $\eta_h(x,t)$ and $\eta_\rho(x,t)$. They will depend on the physical situation under consideration. We shall often take them as two independent Gaussian white noises, characterized by their widths Δ_h, Δ_ρ defined according to

$$\langle \eta_h(x,t)\eta_h(x',t')\rangle=\Delta_h^2\delta(x-x')\delta(t-t'),$$
$$\langle \eta_\rho(x,t)\eta_\rho(x',t')\rangle=\Delta_\rho^2\delta(x-x')\delta(t-t') . \tag{2.6}$$

Thus pouring grains onto a sandpile should be represented by only noise in ρ; alternatively one might imagine that the sandpile is being subjected to vibration at its base, the chief effect of which is transmitted to the surface clusters via the bulk, which hence have only noise in h. For the moment, we consider both noise terms to be present.

These equations contain the first local and analytical formulation of a model presented much earlier [15], in their explicit demonstration of the competition and cooperation between independent-particle and collective dynamics in a sandpile subjected to perturbation; the ideas in that model, however, have been analyzed systematically by other numerical and theoretical methods for a number of years [15,16,19,20,22]. As expected, apart from exceptional regimes where one or the other dominates, these mechanisms have an explicit coupling which is encapsulated in the transfer terms T, originating in the fact that currents of flowing grains must undergo a constant exchange of particles with the clusters in the boundary layer [15].

To close this presentation of our dynamical equations, we briefly present their explicit solution in the absence of noise. Let the initial situation be that of a uniform slope $p=-p_c+\varepsilon$ of immobile grains: $y(x,0)=px$, i.e., $h(x,0)=\varepsilon x$, together with a constant density of mobile particles: $\rho(x,0)=\rho_0$. We thus have a uniform $\nabla h=\varepsilon$. In the subsequent evolution this slope remains constant, and the amounts of mobile and immobile particles are related by the conservation law $h(x,t)=h(x,0)+\rho_0-\rho(t)$. The evolution of $\rho(t)$ crucially depends on the sign of ε, as follows.

(i) Subcritical case ($|p|<p_c$, i.e., $\varepsilon>0$): the density of mobile particles relaxes exponentially to zero, according to

$$\rho(t)=\rho_0 e^{-(\lambda+\mu)\varepsilon t} . \tag{2.7a}$$

(ii) Supercritical case ($|p|>p_c$, i.e., $\varepsilon<0$): the density of mobile particle diverges exponentially, according to

$$\rho(t)=-\rho_1+(\rho_0+\rho_1)e^{\mu|\varepsilon|t} , \tag{2.7b}$$

with $\rho_1=\nu/\mu$.

(iii) Critical case ($|p|=p_c$, i.e., $\varepsilon=0$): the system is entirely frozen:

$$\rho(t)=\rho_0 . \tag{2.7c}$$

The dynamics in the critical case is thus driven by the fluctuations generated by the noisy source terms. It is worth noting that the characteristic times associated with the relaxation law (2.7a) and with the law of divergence (2.7b) both diverge as $\tau\sim 1/|\varepsilon|$, as the critical slope $\varepsilon=0$ is approached.

III. CRITICAL BEHAVIOR

This central section is devoted to an investigation of the rich variety of behavior shown by our dynamical equations in the presence of noise. Let us emphasize the following. First, we do not need to introduce overly complicated terms to induce a change in universality class; a simple, physically motivated description of the physics of clusters and grains on a flowing sandpile suffices to give us critical exponents. At a deeper level this indicates that we have added physics to the linear equation for h (EW) which takes us away from the trivial fixed point, and confirms that our nonlinear decorations of the EW equation as well as the coupling we have introduced between h and ρ is physically meaningful.

Next, it turns out that the most important ingredient in our dynamical equations is the tilt term $\nu(\nabla h)_-$ describing the physics of tilting a pile so that, quite simply, clusters of grains which appear frozen and stationary when the pile is horizontal release grains which flow down the pile when tilted. However, this is not the only term that changes universality classes. For instance, as discussed below, the effect of symmetry between x and $-x$, and different combinations of the noise terms induce changes of universality class. These phenomena will be discussed under the relevant headings of this section; the corresponding estimated critical exponents are listed in Table I and shown in Fig. 3.

More specifically, in order to obtain an idea of which terms of our equations are the most relevant, in the sense of the renormalization-group approach to critical phenomena, we can use the arguments of dimensional

TABLE I. Critical exponents α, β, and z for both fields $h(x,t)$ and $\rho(x,t)$, measured from numerical simulations in the various cases described in text. The exact EW and KPZ values are recalled for comparison. The asterisk means that the exponent z cannot be accurately evaluated from the available data on α and β.

Model	Species	α	β	z
EW	h	$\frac{1}{2}$	$\frac{1}{4}$	2
KPZ	h	$\frac{1}{2}$	$\frac{1}{3}$	3/2
(1) asymmetric	h	0.94±0.07	0.43±0.04	2.2±0.4
noise in h	ρ	0.22±0.08	0.07±0.02	*
(1a) no-tilt	h	0.36±0.02	0.41±0.08	0.9±0.2
noise in h and in ρ	ρ	0.39±0.05	0.27±0.04	1.4±0.4
(2) asymmetric	h	0.80±0.06	0.42±0.07	1.9±0.4
noise in ρ	ρ	0.33±0.10	0.19±0.04	1.7±0.9
(3) asymmetric	h	0.97±0.07	0.45±0.03	2.2±0.3
noise in h and in ρ	ρ	0.12±0.08	0.07±0.03	*
(4) symmetric	h	0.40±0.06	0.37±0.04	1.1±0.3
noise in h				
(5) symmetric	ρ			2
noise in ρ		$\frac{1}{2}$	$\frac{1}{4}$	
(6) symmetric	h	0.37±0.05	0.39±0.05	1.0±0.2
noise in h and in ρ				

analysis referred to as power counting [23]. For the sake of clarity, we first perform this perturbative analysis on the KPZ equation,

$$\partial h/\partial t = D\nabla^2 h + g(\nabla h)^2 + \eta(\mathbf{x},t) \ . \tag{3.1}$$

We rescale space according to $\mathbf{x}\to b\mathbf{x}$, and time according to $t\to b^z t$, with z being the unknown dynamical exponent. We make the natural hypothesis that the noise $\eta(\mathbf{x},t)$ is dimensionless, in the sense that its width Δ, defined in analogy with Eq. (2.6), is independent of b. We can then determine iteratively the power of the linear scaling factor b which affects every quantity or parameter of the model under rescaling, in any dimension d. By definition, the negative of this power is called the (classical or naive) dimension of the quantity under consideration. Starting from the dimensions $[\mathbf{x}]=-1$, $[t]=-z$, and $[\Delta]=0$, we obtain

$$[h]=(d-z)/2, \quad [D]=z-2, \quad [g]=(3z-d-4)/2 \ . \tag{3.2}$$

The general ideas of the renormalization-group approach imply that an operator is relevant (irrelevant) whenever the dimension of the associated coupling constant is positive (negative).

(i) Consider first the linear (EW) theory, obtained for $g=0$. This linear theory is scale invariant for $[D]=0$, thus $z=2$. We thus recover the known scaling properties of the EW theory, in particular $[h]=(d-2)/2$, thus the EW exponents $\alpha=[h]/[x]=(2-d)/2$, and $\beta=[h]/[t]=(2-d)/4$.

(ii) Consider now the full KPZ theory. With $z=2$ we have $[g]=(2-d)/2$, implying that a weak nonlinearity is irrelevant for $d>2$, and relevant for $d<2$, so that the perturbative critical dimension reads $d_c=2$. These predictions agree with the known phase diagram of the KPZ problem, with its weak-fluctuation fixed point for $d<2$, and its nonperturbative scaling behavior for $d\geq 2$. This kind of power-counting analysis is, however, unable to predict that the KPZ exponents are nontrivial in any finite dimension, due to nonperturbative effects [9,10,24].

The power-counting analysis of our dynamical equations (2.2) goes as follows. Under the assumption that the noise is again dimensionless, starting from $[x]=-1$, $[t]=-z$, and $[\Delta_h]=[\Delta_\rho]=0$, we obtain

$$\begin{aligned}&[h]=[\rho]=(d-z)/2, [D_h]=[D_\rho]=z-2 \ ,\\ &[\nu]=z-1, \quad [\lambda]=[\mu]=(3z-d-2)/2 \ ,\\ &[\kappa]=[\gamma]=(3z-d-4)/2 \ .\end{aligned} \tag{3.3}$$

(a) The pseudolinear theory obtained for $\kappa=\lambda=\mu=\gamma=0$ is formally scale invariant if we set $z=1$. We then have a mean-field-like scaling with $[\nu]=0$, where diffusion is irrelevant, since $[D_h]=[D_\rho]=-1$. The other mean-field exponents read $\alpha_h=\beta_h=[h]/[x]=(1-d)/2$, and similarly for ρ.

(b) The fully theory then has critical dimension $d_c=1$, where we have $[\lambda]=[\mu]=0$ (marginal) and $[\kappa]=[\gamma]=[D_h]=[D_\rho]=-1$ (irrelevant).

Perturbation power counting is even more questionable in the present case than for the KPZ problem, since our pseudolinear dynamical equations are already fully nonlinear. The mean-field-like scaling exponents mentioned above may well not be observable in any dimension. It nevertheless provides a hint concerning which perturbations are likely to be the most relevant ones.

We now discuss some general information about the numerical simulations which yield the results discussed below. The simulations have been performed by discretizing Eqs. (2.2) both in space and time. Since we are mainly interested in critical behavior, we have set the step in the spatial direction, i.e., the lattice spacing, equal to unity ($\delta x=1$). We have employed a finite system of size L lattice points, with periodic boundary conditions. The discretization in the temporal direction requires more care, because of the strong instabilities which are intrinsically present in nonlinear growth equations in discrete time. Just as previous authors [25], we have had to use currently small values of the time step, of order $\delta t=10^{-3}$, in order to avoid instabilities, and to generate physically acceptable, well behaved, and stable solutions to our equations.

For each of the cases detailed below, we have calculated the exponent of both h and ρ at criticality. The exponents α and β are *a priori* different for both species, whereas a common value of the dynamical exponent z is expected. The actual evaluation of the exponents has been done by means of the structure factors $S_h(q,\omega)$ and $S_\rho(q,\omega)$, which are defined as the double Fourier transforms of the correlation functions $G_h(x-x',t-t')=\langle h(x,t)h(x',t')\rangle$ and $G_\rho(x-x',t-t')=\langle\rho(x,t)\rho(x',t')\rangle$. The scaling laws (1.2) and (1.1), recalled in the Introduction, can be recast, respectively, as

$$S(q,\omega)\approx\omega^{-1}q^{-1-2\alpha}\Phi(\omega/q^z) \tag{3.4}$$

for small q and ω, and especially

$$\begin{aligned}&S(q,0)\sim q^{-1-2\alpha} \quad (q\to 0) \ ,\\ &S(0,\omega)\sim\omega^{-1-2\beta} \quad (\omega\to 0) \ .\end{aligned} \tag{3.5}$$

A. Asymmetric situation (cases 1–3)

This situation is the most commonly encountered one, of a (sloping) sandpile with a preferred direction of flow; our original presentation of these equations concerned this case [1], and the pertinent equations are Eqs. (2.2). We refer to it as the asymmetric case because it describes the physics of fluctuations with respect to a uniform slope $-p_c$, so that there is no $x\leftrightarrow -x$ symmetry.

In numerical simulations we have set the irrelevant couplings γ and κ equal to zero, and chosen $D_h=D_\rho=1$. The transfer term T deserves some more attention. The effect of the term $\lambda\rho(\nabla h)_+$, as mentioned above, is to limit the fluctuations of ρ around its finite mean value of order $\langle\rho\rangle\sim\nu/\lambda$. Critical fluctuations can therefore only develop for $\nu\gg\lambda$. We choose to set $\lambda=1$ and $\mu=0$ (again for simplicity), keeping $\nu\gg 1$ as a free parameter, besides Δ_h and Δ_ρ. In practice, values of ν of order 10–50 turn out to be large enough in order not to alter the critical fluctuations, for sizes ($L\leq 1000$) and observation times ($t \leq 10^5$) actually used in our numerical simulations.

The dynamical equations thus read

$$\text{Cases 1–3:} \begin{cases} \partial h/\partial t = \nabla^2 h - T + \eta_h(x,t) \\ \partial \rho/\partial t = \nabla^2 \rho + T + \eta_\rho(x,t) \\ T = -\rho(\nabla h)_+ - \nu(\nabla h)_- \ . \end{cases} \quad (3.6)$$

We have considered the following three cases, according to the nature of the noise: (i) Case 1: noise in h ($\Delta_h > 0, \Delta_\rho = 0$). (ii) Case 2: noise in ρ ($\Delta_h = 0, \Delta_\rho > 0$). (iii) Case 3: noise in h and ρ ($\Delta_h > 0, \Delta_\rho > 0$).

Nontrivial long-range spatial and temporal critical fluctuations are observed for both species h and ρ in these three cases, as illustrated for case 1 on Figs. 1 and 2. Figure 1 shows log-log plots of the spatial structure factors against reduced wave vector $q/(2\pi)$, whereas Fig. 2 shows log-log plots of the temporal structure factors against frequency $\omega/(2\pi)$. In each case we have estimated the critical exponents α and β for both species by fitting log-log plots of the structure factors (those presented in Figs. 1 and 2 and similar ones for the other cases) to straight lines, according to Eq. (3.5). The data for several different system sizes and observation times are well fitted to a single power law. Table I gives our estimates for the critical exponents, as well as estimates for the corresponding error bars, incorporating statistical errors as well as systematic ones, the latter being necessarily evaluated in a rather subjective way. Figure 3 shows a scatter plot in the α-β plane of the numerical values of all these critical exponents.

We postpone the general analysis of these exponents to Sec. IV, and now discuss the specific cases. Case 1 is of particular interest, as it is the most directly comparable with EW and KPZ. We observe a rougher behavior of the h profile than the two aforenamed. This effect is very pronounced in the spatial direction ($\alpha_h \approx 0.94$ is to be compared with $\frac{1}{2}$ in both cases), and still appreciable in the temporal one ($\beta_h \approx 0.45$ is to be compared with $\frac{1}{3}$ and $\frac{1}{4}$. Intuitively, this appears to be due to the roughening effect of the tilt term, whose chief role is to cause a generation of flowing grains at points of excessively high

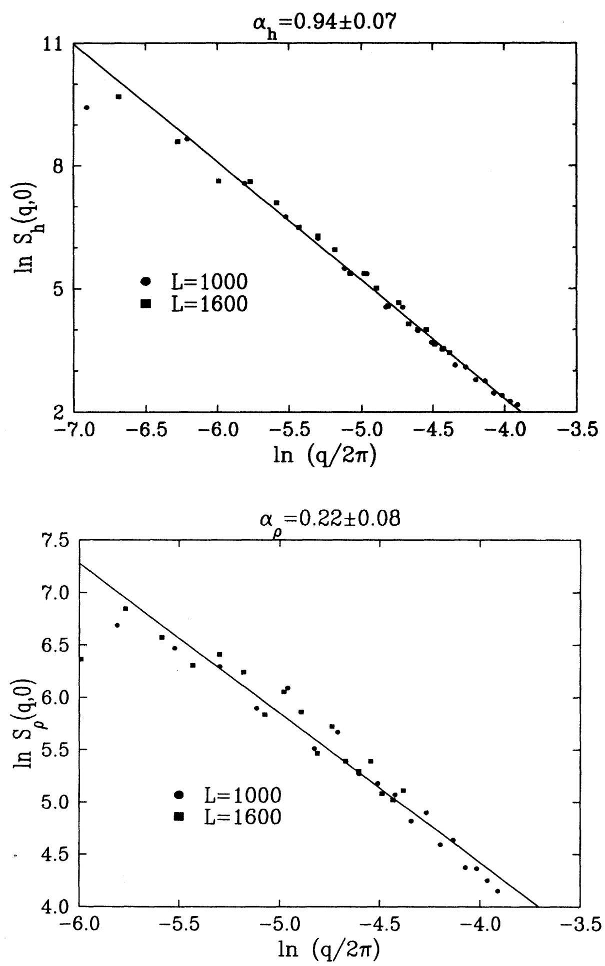

FIG. 1. Log-log plot of the spatial structure factors of case 1, against reduced wave vector $q/(2\pi)$: (a) $S_h(q,0)$. The fitted line has a slope $-1-2\alpha_h = -2.88 \pm 0.14$. (b) $S_\rho(q,0)$. The fitted line has a slope $-1-2\alpha_\rho = -1.44 \pm 0.16$.

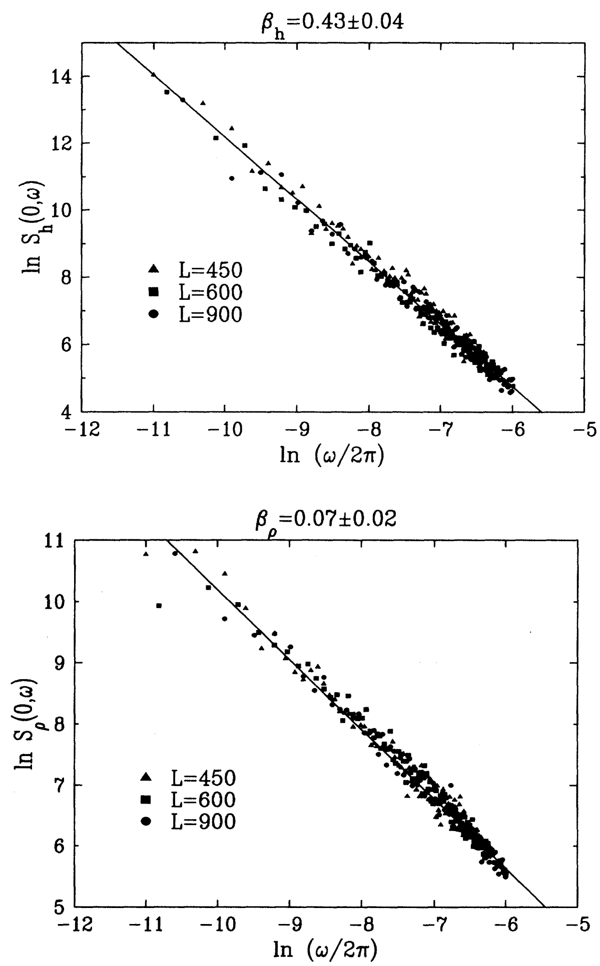

FIG. 2. Log-log plot of the temporal structure factors of case 1, against frequency $\omega/(2\pi)$. (a) $S_h(0,\omega)$. The fitted line has a slope $-1-2\beta_h = -1.86 \pm 0.08$. (b) $S_\rho(0,\omega)$. The fitted line has a slope $-1-2\beta_\rho = -1.14 \pm 0.04$.

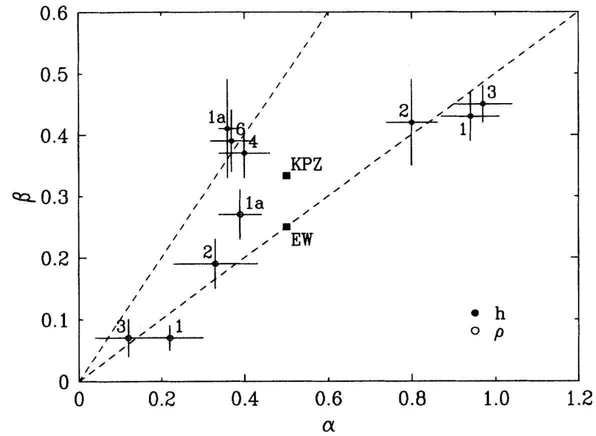

FIG. 3. Plot of the numerical values of the critical exponents listed in Table I in the α-β plane. The lines $z=\alpha/\beta=1$ and $z=\alpha/\beta=2$ are meant as guides to the eye.

slope on the surface. The effect is thus of an enhanced deposition, so that we expect the values of the exponent β_h to lie between $\frac{1}{4}$ (deposition and relaxation) and $\frac{1}{2}$ (deposition only, no relaxation). This is indeed what we have measured. However, given that the roughening effect appears to manifest itself more directly in the temporal exponent, since $\alpha^{\rm KPZ}=\alpha^{\rm EW}$ but $\beta^{\rm KPZ}>\beta^{\rm EW}$, we will restrict ourselves to discussing it in this context.

In order to check further the validity of the above intuitive idea, we compare β_h in this case with that of the model where the tilt term is replaced by a noise in ρ. In this situation, referred to as case 1a, the dynamical equations read

$$\text{Case 1a:}\begin{cases}\partial h/\partial t=\nabla^2 h-T+\eta_h(x,t)\\ \partial\rho/\partial t=\nabla^2\rho+T+\eta_\rho(x,t)\\ T=-\mu\rho\nabla h\ .\end{cases}\tag{3.7}$$

We find that, indeed, β_h is unchanged, within error bars, with respect to case 1 (and, as it turns out, cases 2 and 3 as well) verifying that the role of the tilt term is analogous to that of a local generator of flowing grains, i.e., a noise in ρ.

We mention that, for the purpose of numerical stability, we have actually replaced the transfer term T of Eq. (3.7) by an odd nonlinear function $T'=F(T)$ of the same quantity, linear at small T and saturating at large T; we have chosen

$$F(T)=\begin{cases}-1 & \text{for } T\le -1\\ T & \text{for } -1\le T\le 1\\ 1 & \text{for } T\ge 1\end{cases}\tag{3.8}$$

in order to achieve a fast numerical evaluation.

Case 2 describes the situation of a noise in ρ, in the presence of tilt. This corresponds to the case of pouring grains onto a sandpile, whereas case 1 corresponded to shaking one. In order to initiate nontrivial dynamics, we have to subject the pile to a transient noise in h for a period $(0<t<t_0)$. The exponents in this case show a small, although perhaps significant, difference from the previous case. The ρ exponents are slightly larger than previously, while the h exponents are smaller than in case 1. Intuitively, this should be expected, since in this case the major perturbation corresponds to a deposition of flowing grains. However, and somewhat surprisingly, we still see that the h exponents in this case are larger than the corresponding ρ exponents. Our tentative explanation for this is that, whatever the nature of the noise, the transfer term T predominates; thus, while the difference in the noise terms accounts for the relative values of, say β_ρ, in cases 1 and 2, the nonlinear couplings in T ensure that, in any given case, $\beta_h>\beta_\rho$. Another way of saying this is that our dynamical equations result in a sandpile surface whose roughness is due more to embedded clusters than flowing grains, which is clearly sensible; also, a comparison of cases 1 and 2 shows that shaking the pile is a more efficient way of generating roughness than pouring grains down it, which also seems intuitively plausible.

These speculations are reinforced to some extent by case 3, which corresponds to noises in h and ρ. Within error bars, we observe no significant difference between this case and case 1 (noise in h alone), indicating that noise in ρ does not materially affect the situation of a noise in h in the presence of the tilt term. In other words, once we have generated the maximal roughness of the pile by shaking its clusters, we will not materially change the observed roughness of the surface by, in addition, depositing grains on it.

We have so far expressed all the surface heights as deviations from one unique critical slope $-p_c$. However, we know [1–3] that there is in reality a range of angles of repose for a sandpile, and we should have included these to describe the different thresholds for the onset and continuation of avalanches. We propose to do this via the introduction of a threshold C, both in the tilt and current terms. The incorporation of two thresholds p_c and C in our equations models the situation in real sandpiles, where the former corresponds to the minimum angle of repose (onset of avalanches), and the latter to the maximum angle of stability (threshold for continuous avalanching). We thus write

$$\begin{aligned}&\partial h/\partial t=D_h\nabla^2 h-T+\eta_h(x,t)\ ,\\ &\partial\rho/\partial t=-\nabla j+D_\rho\nabla^2\rho+T+\eta_\rho(x,t)\ ,\\ &T=-\kappa\rho\nabla^2 h-\lambda\rho(\nabla h+C)_+-\mu\rho\nabla h-\nu(\nabla h)_-\ ,\\ &j=-\gamma\rho(\nabla h+C)_-\ .\end{aligned}\tag{3.9}$$

The threshold C can be viewed as an extra coupling constant; its dimensional content as dictated by power counting reads $[C]=(d-z+2)/2=(d+1)/2$ with the mean-field-like dynamical exponent $z=1$. The introduction of a nonzero threshold thus drives the system away from criticality, at least within the perturbative power-counting approach. This prediction has been confirmed by numerical simulations, which show a definite crossover to noncritical fluctuations when C is switched on in a progressive way.

From a more physical point of view, this implies that the introduction of the second threshold C takes our system away from the critical behavior of a second-order phase transition to the more realistic first-order behavior characteristic of real sandpiles [1–3]. Our choice of the simplified critical equations in this paper was made for the simple purpose of demonstrating the critical exponent we obtain, and so showing the validity of the physics we have included; this being done, we emphasize that in any experimental application, the full equations (3.9) with two thresholds should be used.

B. Symmetric situation (cases 4–6)

We now turn to the analysis of a version of our dynamical equations which has the $x \leftrightarrow -x$ symmetry. This can be visualized as the surface of a sandpile which is flat on average, and is subject as before to deposition and/or shaking. Physically this represents an important difference in that flow is allowed in both directions, unlike in the previous case of a background slope, where gravity imposes a preferred direction for grain flow. The dynamical equations in full generality read

$$\text{Cases 4–6:} \begin{cases} \partial h/\partial t = D_h \nabla^2 h - T + \eta_h(x,t) \\ \partial \rho/\partial t = -\nabla j + D_\rho \nabla^2 \rho + T + \eta_\rho(x,t) \\ T = -\kappa\rho\nabla^2 h - \lambda\rho|\nabla h| + \nu(|\nabla h| - C)_+ \\ j = \gamma\rho\,\mathrm{sgn}(\nabla h)(|\nabla h| - C)_+ \; . \end{cases} \tag{3.10}$$

In order to investigate the critical regime, we choose to simplify the above equations for $\gamma = C = 0$ as

$$\begin{aligned} \partial h/\partial t &= \nabla^2 h - T + \eta_h(x,t) \; , \\ \partial \rho/\partial t &= \nabla^2 \rho + T + \eta_\rho(x,t) \; , \\ T &= -\rho\nabla^2 h + (\nu - \rho)|\nabla h| \; . \end{aligned} \tag{3.11}$$

An inspection of Eq. (3.10) shows the effect of symmetrizing the tilt term; a finite value of the local slope now indicates a deviation from the equilibrium flat surface, so that it is the absolute magnitude of the local slope which results in immobile clusters being converted to flowing grains. This symmetrization applies equally to the other term in T, which arranges for cluster-grain exchange depending on $\nabla^2 h$ rather than ∇h. We again consider three cases, according to the number of components of the noise: (iv) Case 4: noise in h ($\Delta_h > 0, \Delta_\rho = 0$). (v) Case 5: noise in ρ ($\Delta_h = 0, \Delta_\rho > 0$). (vi) Case 6: noise in h and in ρ ($\Delta_h > 0, \Delta_\rho > 0$).

In Cases 4 and 6, we observe $\alpha_h \approx \beta_h \approx 0.40$ and $z_h = 1$ within error bars, just as in case 1a, whereas $\rho(x,t)$ does not exhibit divergent fluctuations: the structure factors $S(q,0)$ and $S(0,\omega)$ rather saturate to constant values for small enough wave vector q or frequency ω, implying the decay of ρ-ρ correlations at long separations in space and in time.

These results show that the dynamics corresponding to anisotropic and isotropic sandpiles are quite different. Our interpretation of these results is based on the symmetry of the pile. While the tilt term still performs its earlier role with respect to β_h (which is identical within error bars to cases 1–3), there is an important difference between what happens to the flowing grains once they are released from their erstwhile clusters. With no preferred direction of flow, we would expect backflow of grains on a regular basis, contrary to the anisotropic case; thus we should expect that different clumps of ρ generated by possibly anticorrelated bursts of j should have decaying correlations in space and time. A way of verifying this conjecture consists in checking j-j correlations in this case and to compare with the asymmetric case; we have done this and found, as expected, that j is well correlated in the asymmetric case (case 1) (corresponding to unidirectional flow), but is weakly correlated (cases 4 and 6) in the symmetric case.

Another interesting situation is case 5, where noise is present only in the equation for ρ. Just as for case 2, we have to put noise into h too in an initial period. We find that h becomes frozen, soon after the initial period has elapsed, into a nontrivial rough landscape, entirely inherited from the transient period, and therefore characterized by a roughness exponent $\alpha \approx 0.40$. The evolution of ρ which then takes place, with this frozen height configuration as a background, is effectively linear, and the EW exponents are observed, in spite of the background. This is actually to be expected by simple physical reasoning: the effect of pouring flowing grains on a flat surface (which has no preferred direction of flow for the flowing grains ρ) will, after transients, have relatively little effect on h. In the absence of this coupling, ∇h rapidly approaches a frozen profile across the surface, so that the tilt term becomes inactive; the only remaining effect is that buildups of flowing grains are gradually diffused away across the frozen h landscape, leading to the observed EW exponent of $\beta_\rho = \frac{1}{4}$. Note that things would have been quite different if we had had a coupling term involving $\nabla\rho$.

IV. DISCUSSION

Before discussing the essential points of our results, we review the original motivation for this work. Until the recent spate of interest in sandpiles, the theoretical study of evolving interfaces has (correctly) focused on dynamical equations which involved only one variable, the local height of the surface $h(\mathbf{x},t)$ [4,6,7].

Early work on sandpile dynamics, however, showed the necessity for a grain-cluster coupling; in particular, a microscopic model was put forward [15], encapsulating the competition and cooperation between independent-particle (mobile grains) and collective (clusters or relatively immobile grains) dynamics in a sandpile subjected to perturbation. The main predictions of that model were based on the qualititatively and quantitatively different effects of the two kinds of dynamics on material properties like the structure; for instance, the faster (independent-particle) dynamics was expected to lead to less compact structures which were formed relatively quickly, whereas the slower (collective) dynamics was postulated to lead to denser, more stable structures which took longer to form. These ideas were tested by a combination of numerical and theoretical methods for a number of years [15,16], and the results of these investigations were found to be in good agreement with experiments [2,3].

Other concurrent developments involved an investigation of evolving configurational disorder [19] and the effect of tilt [20]; these further tested the ideas of cluster reorganizations, demonstrated the configurational hysteresis characteristic of these systems [2,3], and showed the effects of dissipation and inertia. Most crucially, these investigations provided a quantitative basis for what had till then been at the level of intuition: the importance of tilt in the cluster-grain exchange process.

These tests accomplished, all of these concepts were given a local and analytical form, and the first such coupled nonlinear equations involving a cluster-grain exchange process were published in a generalized form in Ref. [1]. We based our choice of the collective coordinate on earlier work, since the variations of local height $h(\mathbf{x},t)$ clearly represent configurational fluctuations caused by changes in cluster shapes and sizes. The coordinate representing independent particles was chosen to be its logical embodiment, the local density $\rho(\mathbf{x},t)$ of mobile particles.

The uncoupled terms were chosen so as to represent (i) pure cluster reorganization (strictly collective relaxation), and (ii) the reorganization of flowing grains (strictly independent-particle relaxation).

The coupled terms were chosen to represent (i) intercluster motion initiated by the flowing grains (slope independent, a representation of inertia), (ii) intercluster motion mediated by the flowing grains (slope dependent, representation of evolving configurational disorder and memory), (iii) spontaneous generation of flowing grains for larger-than-critical local slopes (effect of tilt), and (iv) the saturating term for flowing grains released by tilt (effect of boundary layer).

Our choice of perturbation was likewise general, incorporating noise in ρ as well as h. In this paper, we have concentrated on the case of white noise in both species, and have presented results for each one in isolation, as well as in combination with the other. We have thus focused on the cases of uniformly poured, shaken, and poured *and* shaken sandpiles. It is worth pointing out, however, that a simple change in noise characteristics will generate the case of, say, a sandpile with grains being poured at a point, or, for that matter, a sandpile being drained either uniformly or at a point. We leave the detailed exploration of these cases, as well as the investigation of the effect of different boundary conditions, to future work.

In the absence of noise, our dynamical equations showed the expected behavior: when the slope was below critical, the current of flowing grains died down, and their density relaxed exponentially to zero. When the slope was above critical, the density of mobile particles exploded exponentially. In other words, as we would expect, a finite number of mobile grains will, in a supercritical pile, be able to augment, via the tilt and exchange terms; in a subcritical pile, on the other hand, any initially mobile grains will soon find suitable voids to occupy them, thus becoming part of the clusters in the boundary layer. The critical nature of the critical slope is clearly apparent from the divergence of the characteristic times associated with relaxation or explosion, both above and below criticality.

The most detailed investigations in our paper concerned the critical state in the presence of noise. We found nontrivial critical exponents in a number of different cases; this convinced us that our phenomenological attempts to model sandpile surfaces had succeeded in adding physics to the linear equations for h in that our leading nonlinearities have been found to be relevant. Figure 3 demonstrates an evident clustering of the exponents concerning h. This observation leads us to the hypothesis that there is, in a first approximation, a single independent exponent $\alpha^{\text{tilt}} \approx 0.40$, such that we have

$$\begin{aligned} &\text{Cases 1, 2, and 3:}\quad \alpha_h \approx 2\alpha^{\text{tilt}},\quad \beta_h \approx \alpha^{\text{tilt}},\quad z\approx 2,\\ &\text{Cases 1a, 4, and 6:}\quad \alpha_h \approx \beta_h \approx \alpha^{\text{tilt}},\quad z\approx 1\;. \end{aligned} \tag{4.1}$$

We definitely cannot rule out the existence of a fine structure, i.e., of small differences between the exponents of various cases in each of the two broad universality classes mentioned above. The situation of the exponents related to ρ is less clear. Furthermore, we are unable to compare the nontrivial values of exponents with predictions existing in the literature, since, to our knowledge, no similar coupled nonlinear equations, which are nontrivial in their criticality behavior and correspond directly to similar physical situations, have so far been investigated.

In case 1, which was most directly comparable with EW and KPZ, that of a sandpile on a slope in the presence of vibration, we observed a roughening of the surface compared to the two above. We speculated that this could be due to the roughening effect of the tilt term which acts as a local generator of flowing grains in areas of excess slope; and crudely tested this hypothesis by comparing the temporal exponent β_h with that obtained in case 1a (coupled equations in the absence of tilt, with two noises). That verified, we feel justified in claiming that the presence of tilt in this case is responsible for obtaining rougher surfaces than in either EW or KPZ.

Let us examine the implications of cases 2 and 3 next: everyday experience tells us that if we want to create a pile with a rough surface, shaking would be a more efficient way to do this compared to pouring. The reason is that shaking directly affects cluster shapes on the bulk and surface, whereas pouring would in general tend to cause grains to flow down the pile. In the parlance of previous work [1], a noise in h predominantly affects the collective motion of clusters, whereas a noise in ρ predominantly affects the motion of independent grains. Also, since clusters are in general able to sustain roughness more than moving grains, we would expect that the roughness exponents of the former would exceed those of the latter species. We indeed find that, in all of the cases of interest (1–3), $\beta_h > \beta_\rho$; and that in case 2 (corresponding to a noise in ρ alone), while the roughness of the h landscape is suppressed, the roughness of the ρ landscape is not overwhelmingly enhanced. Also, cases 1 and 3 are not materially different, showing that the process of deposition of flowing grains does not make much difference to the roughness profile of a sandpile that has already been subjected to shaking. It is remarkable that, although we have in no sense built in these features into our equations, we appear to obtain a physically desirable and meaningful behavior from them.

Let us emphasize that in the discussion above we refer only to temporal roughening, as will be obvious from the fact that we have discussed only the β exponents in all the cases. For instance, the effect of the tilt term in case 1 is to generate a surface whose correlations *in time* are greatly enhanced compared to EW or KPZ. The case of the α exponents, i.e., the issue of spatial correlations as a function of distance, is altogether more complex and subtle. For example, there is a considerable controversy regarding whether sandpiles are asymptotically rough or smooth in the spatial sense: proponents of smooth surfaces argue that sand dunes in deserts are extremely smooth, and theoretical work by Hwa and Kardar [12] support this claim. On the other a hand, we obtain large values of α; also, Bouchaud *et al.* [21] find that their surfaces are characterized by the trivial exponents of the linear EW theory, leading them to conclude that their surfaces are asymptotically rough. In addition, our results for case 1 show remarkable agreement with experimental work [26] on rotated and shaken sandpiles, where measured exponents of $\alpha_h = 0.92 \pm 0.05$ and $\beta_h = 0.48 \pm 0.16$ are in close agreement with case 1 of Table I. Our tentative suggestion is that this controversy may be resolved if more careful consideration is given to the *other* processes at work on the sandpiles concerned; for instance, in a windless desert, dunes may tend to manifest relatively little roughness, whereas the presence of wind or another perturbing fluid would cause the dynamical effects inherent in our equations to predominate and cause greater spatial roughness.

So far, the universality classes we have obtained were for a sloping pile, corresponding to different combinations of noise. We felt that, in analogy with other critical phenomena, symmetry could also be an important factor in determining universality classes. For this reason we examined the effect of deposition and/or shaking on a *flat* surface, where the important physical difference is that the bidirectional flow of mobile grains is allowed in the absence of a biasing slope. One would expect intuitively that, in the absence of noise, this would lead to the ρ correlations becoming much weaker, if not disappearing altogether. Cases 4 and 6 show that this is indeed the case. In case 5, the absence of a noise in h causes the h profile to stop fluctuating after a time; the tilt term becomes inactive, and the noise in ρ generates a diffusive response from the flowing grains across the frozen (but still rough) h landscape.

Next, we discuss the issue of thresholds, known to be important for the hysteretic and bistable behavior of sandpiles [2,3]. It is well known that sandpile dynamics shows behavior which is more characteristic of a first-order phase transition than the critical behavior corresponding to a second-order phase transition [1–3]. This is why our dynamical equations (3.9) were written down with two thresholds, one corresponding to the onset (minimum angle of repose) and the other corresponding to the continuation (maximum angle of stability) of avalanches. However, given the phenomenological nature of our work, we wanted to provide hard evidence that the nonlinear decorations we had added to our linear coupled equation were meaningful physical additions, rather than vacuous and needless complications. It was for this purpose that we simplified Eqs. (3.9) to their second-order form (2.2) to look for critical exponents which we have presented in this paper. This done, we emphasize our preference for the first-order form of the equations in terms of modeling physical reality.

Before summing up, we return to the subject of tilt, which has formed a *leitmotif* for our work. Although it might appear from our stack of critical exponents and their obvious connection to the tilt term that this was a sufficient *raison d'être* for the latter, we emphasize that a very important feature resulting from its inclusion has to do with experimental work. The rotating cylinder apparatus has been used for a very long time [27] to investigate sandpile dynamics, and its use remains popular with experimentalists to this day [3]. Clearly, the driving force in this setup is the constant tilt to which the sand in the cylinder is subjected, so that a modeling of the effect of tilt is essential in any equations which seek to interpret such experiments. For us, the successful inclusion of tilt in our dynamical equations brings with it the enormous potential benefit of being of use in interpreting such a massive body of experimental work.

Finally, we summarize our work. We have presented a set of phenomenological equations to model the dynamics of sandpile surfaces. These include nonlinear couplings to represent the constant transfer between relatively immobile clusters and mobile grains, incorporate the presence of tilt, and contain representations of inertia and evolving configurational disorder, which a previous body of work [1] has shown to be important. We have looked at the response of these equations to different perturbations, and presented our critical exponents for these, both in the presence and absence of a biasing slope. It is our hope that this work will lead to further detailed experimental and theoretical investigations in this very exciting and topical field.

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